

# Intertwining Operators and Quantum Homogeneous Spaces

Leonid L. Vaksman <sup>1</sup>

Physico-Technical Institute of Low Temperatures, Kharkiv, Ukraine

**Abstract.** In the present paper the algebras of functions on quantum homogeneous spaces are studied. The author introduces the algebras of kernels of intertwining integral operators and constructs quantum analogues of the Poisson and Radon transforms for some quantum homogeneous spaces. Some applications and the relation to  $q$ -special functions are discussed.

## 1 Introduction

The integral transformations which intertwine quasi-regular representations of groups are widely used in representation theory, harmonic analysis, and mathematical physics. The kernels of the intertwining integral operators are functions on Cartesian products  $X \times Y$  of  $G$ -spaces which are constant on the  $G$ -orbits. These can be distributions, as in the case of the Radon transform

$$K(x, y) = \delta(c - (x, y)) \quad (1.1)$$

and other operators related to the integral geometry.

The quantum group theory originated in the papers by V.G.Drinfeld and M.Jimbo [2, 5] in the middle eighties under the influence of the quantum inverse scattering method developed by L.D.Faddeev and his school.

Our goal is to construct invariant kernels analogous to (1.1) and the corresponding intertwining integral operators for a certain class of quantum groups and their homogeneous spaces. At the same time there will be defined real powers of the  $q$ -Poisson kernel and shown their relation to the theory of  $q$ -special functions. (Here  $q \in (0, 1)$  is a parameter related to the "Plank constant"  $\hbar$  used in [2] by the equality  $q = e^{-\hbar/2}$ .)

Quantum algebras of functions are deformations of algebras of functions on Poisson manifolds. Given two Poisson manifolds  $X$  and  $Y$ , the Poisson bracket on the Cartesian product  $X \times Y$  can be defined by

$$\{\varphi_1 \otimes \psi_1, \varphi_2 \otimes \psi_2\} = \{\varphi_1, \varphi_2\} \otimes \psi_1 \psi_2 - \varphi_1 \varphi_2 \otimes \{\psi_1, \psi_2\}$$

where  $\varphi_i \in C^\infty(X)$ ,  $\psi_i \in C^\infty(Y)$  ( $i = 1, 2$ ). The 'minus' sign is important when  $X$  and  $Y$  are Poisson homogeneous  $G$ -spaces for a Poisson Lie group  $G$  [2]. Due

---

<sup>1</sup>Partially supported by the AMS FSU grant and by the ISF grant.

to it, the Poisson bracket of  $G$ -invariants in  $C^\infty(X \times Y)$  is again a  $G$ -invariant. The quantum analogue of this simple observation is that the intertwining kernels form an algebra (see Section 2). In particular, a polynomial of an intertwining kernel is again an intertwining kernel. However, as we see on the example of (1.1), it does not suffice to have polynomials only, we need distributions. The corresponding constructions are given in Sections 4 and 5.

Finally, it is important to note that the algebras of intertwining kernels are described by extremely simple commutation relations between the generators (see Section 6). This can be explained by the fact that quantization preserves the dimensions of the graded components of the space of intertwining operators.

## 2 Integral Operators and $A$ -module algebras

Consider a Hopf algebra  $A$  over  $\mathbf{C}$ . Recall (cf. [2]) that the comultiplication  $\Delta : A \rightarrow A \otimes A$  is a homomorphism, which allows to introduce the operation of tensor product of  $A$ -modules by. Also, given an  $A$ -module  $V$ , one can define the dual  $A$ -module  $V^*$  by

$$(av^*)(v) = v^*(S(a)v)$$

where  $a \in A$ ,  $v \in V$ ,  $v^* \in V^*$ , and  $S : A \rightarrow A$  is the antipode. The trivial  $A$ -module  $\mathbf{C}$  is defined by the counit  $\varepsilon : A \rightarrow \mathbf{C}$ . By definition, the  $A$ -invariants of an  $A$ -module  $V$  are the vectors  $v \in V$  such that  $av = \varepsilon(a)v$  for any  $a \in A$ . Recall, finally, that the antipode  $S$  is both an algebra and a coalgebra anti-isomorphism of  $A$ .

Important examples of Hopf algebra are the quantum universal enveloping algebras  $U_q \mathfrak{g}$  described in [2, 5]. Given a simple complex Lie algebra  $\mathfrak{g}$  with Cartan matrix  $A = (a_{ij})_{i,j=1}^n$  where  $n = \text{rank } \mathfrak{g}$ , the algebra structure on  $U_q \mathfrak{g}$  is defined by the generators  $X_i^\pm$ ,  $K_i^\pm$  ( $i = 1, \dots, n$ ) and the relations

$$K_i^\pm K_j^\pm = K_j^\pm K_i^\pm, \quad K_i^+ K_i^- = K_i^- K_i^+ = 1, \quad (2.1)$$

$$K_i^+ X_j^\pm = q^{\pm \frac{a_{ij}}{2}} X_j^\pm K_i^+, \quad K_i^- X_j^\pm = q^{\mp \frac{a_{ij}}{2}} X_j^\pm K_i^-, \quad (2.2)$$

$$X_i^+ X_j^- - X_j^- X_i^+ = \delta_{ij} \frac{K_i^{+2} - K_i^{-2}}{q - q^{-1}}, \quad (2.3)$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k q_i^{k(1-a_{ij}-k)} \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i^2} (X_i^\pm)^{1-a_{ij}-k} X_j^\pm (X_i^\pm)^k = 0 \quad (2.4)$$

when  $i \neq j$

where  $q_i = q^{\frac{(\alpha_i, \alpha_i)}{2}}$  (with  $\{\alpha_i\}_{i=1}^n$  being a system of simple coroots). Here  $\begin{bmatrix} m \\ n \end{bmatrix}_t = \frac{(t; t)_m}{(t; t)_n (t; t)_{m-n}}$  is the  $t$ -binomial coefficient,  $(a; t)_k = (1-a)(1-at)\dots(1-at^{k-1})$ .

The coalgebra structure on  $U_q\mathfrak{g}$  is given by

$$\Delta(K_i^\pm) = K_i^\pm \otimes K_i^\pm, \quad \Delta(X_i^\pm) = K_i^\mp \otimes X_i^\pm + X_i^\pm \otimes K_i^\mp \quad (2.5)$$

$$\varepsilon(K_i^\pm) = 1, \quad \varepsilon(X_i^\pm) = 0, \quad S(K_i^\pm) = K_i^\mp, \quad S(X_i^\pm) = -q_i^{\mp 1} X_i^\pm \quad (2.6)$$

**Definition 2.1.** Let  $A$  be a Hopf algebra and  $F$  an algebra equipped with an  $A$ -module structure. We say that  $F$  is an  $A$ -module algebra if the multiplication map

$$m : F \otimes F \rightarrow F, \quad m(f_1 \otimes f_2) = f_1 f_2$$

is a morphism of  $A$ -modules.

As follows from (2.5)-(2.6), in the case of  $A = U_q\mathfrak{g}$  the condition of Definition 2.1 means that

$$K_i^\pm(f_1 f_2) = K_i^\pm(f_1) K_i^\pm(f_2) \quad (2.7)$$

$$X_i^\pm(f_1 f_2) = K_i^\mp(f_1) X_i^\pm(f_2) + X_i^\pm(f_1) K_i^\mp(f_2) \quad (2.8)$$

for any  $i = 1, \dots, n$ ,  $f_1, f_2 \in F$ . The last equality is an analogue of the Leibnitz formula.

Reasonable examples of  $A$ -module algebras – algebras of functions on quantum homogeneous spaces – are given in the paper [3] by L.D.Faddeev, N.Yu.Reshetikhin, L.A.Takhtajan. One more example is the tensor algebra  $T$  generated by  $A$ -modules  $V_1, V_1^*, V_2, V_2^*, \dots, V_k, V_k^*$ . The algebras in [3] are quotients of the tensor algebras over ideals defined in terms of certain solutions of the quantum Yang-Baxter equation.

As in the limit case  $q = 1$ ,  $A$ -module morphisms can be obtained by using the operation of tensor coevaluation (for example,  $V_j^* \otimes V_j \rightarrow \mathbf{C}$ ), and invariants can be obtained by using the canonical embeddings  $\mathbf{C} \hookrightarrow V_j \otimes V_j^*$ . We will need the following well-known properties of  $A$ -modules.

**Proposition 2.2.** Let  $U$  and  $V$  be  $A$ -modules, and  $l : V \otimes U \rightarrow \mathbf{C}$  a morphism of  $A$ -modules. Then  $l(av \otimes u) = l(v \otimes S(a)u)$  for any  $a \in A$ ,  $u \in U$ ,  $v \in V$ .

*Proof.* In the special case when  $V = U^*$  the statement of the proposition follows from the definition of dual module. The general case can be reduced to this special case, since the linear operator  $V \rightarrow U^*$  defined by  $l$  is a morphism of  $A$ -modules:

$$V \simeq V \otimes \mathbf{C} \hookrightarrow V \otimes U \otimes U^* \xrightarrow{l \otimes \text{id}} \mathbf{C} \otimes U^* \simeq U^*$$

**Corollary 2.3.** Let  $F$  be an  $A$ -module algebra and  $\nu : F \rightarrow \mathbf{C}$  a morphism of  $A$ -modules (called invariant integral on  $F$ :  $\int f d\nu \stackrel{\text{def}}{=} \nu(f)$ ). Then for any  $a \in A$  and  $f_1, f_2 \in F$ , the following “integration by parts” formula holds:

$$\int (af_1)f_2 d\nu = \int f_1(S(a)f_2) d\nu \quad (2.9)$$

**Proposition 2.4.** Consider  $A$ -modules  $V_1, V_2$  and the image of  $\text{Hom}_A(V_1, V_2)$  under the canonical isomorphism  $\text{Hom}_{\mathbf{C}}(V_1, V_2) \simeq V_2 \otimes V_1^*$ .

- (i) Every invariant in  $V_2 \otimes V_1^*$  belongs to the image of  $\text{Hom}_A(V_1, V_2)$ .
- (ii) Given an element  $u$  of the image of  $\text{Hom}_A(V_1, V_2)$ , one has that

$$(a \otimes 1)u = (1 \otimes S^{-1}(a))u \quad (2.10)$$

for any  $a \in A$ .

- (iii) If  $u$  satisfies (2.10) for any  $a \in A$ , then  $u$  is an invariant.

*Proof.* If  $u \in V_2 \otimes V_1^*$  is an invariant, then the multiplication by it  $V_1 \rightarrow V_2 \otimes V_1^* \otimes V_1$  with the subsequent coevaluation  $V_2 \otimes V_1^* \otimes V_1 \rightarrow V_2 \otimes \mathbf{C} \simeq V_2$  gives an element of  $\text{Hom}_A(V_1, V_2)$ . This proves the first statement, while the other two follow from the definitions.

**Corollary 2.5.** The space of invariants of  $V_2 \otimes V_1^*$  is determined by the condition (2.10) and canonically isomorphic to  $\text{Hom}_A(V_1, V_2)$ .

Now we are going to give a definition of the algebra of intertwining kernels. To any algebra  $F$  we assign an algebra  $F^{\text{op}}$  obtained from  $F$  by changing the multiplication by the opposite one:  $m^{\text{op}}(f_1 \otimes f_2) = f_2 f_1$ . We have the following proposition.

**Proposition 2.6.** For any  $A$ -module algebras  $F_1$  and  $F_2$ , the invariants in  $F_2 \otimes F_1^{\text{op}}$  form a subalgebra.

*Proof.* By Corollary 2.5, we get a system of equations on the invariants:

$$\forall a \in A : (a \otimes 1)f = 1 \otimes S^{-1}(a)f \quad (2.11)$$

Equip the algebra  $F_2 \otimes F_1^{\text{op}}$  with an  $A \otimes A^{\text{op}}$ -module algebra structure, letting

$$a_2 \otimes a_1 : f_2 \otimes f_1 \mapsto a_2 f_2 \otimes S^{-1}(a_1) f_1$$

for any  $a_1, a_2 \in A$  and  $f_1 \in F_1^{\text{op}}$ ,  $f_2 \in F_2$ . The system (2.11) takes the form

$$(a \otimes 1)f = (1 \otimes a)f$$

It is clear that its solutions form a subalgebra.

**Definition 2.7.** The subalgebra of invariants in  $F_2 \otimes F_1^{\text{op}}$  is called *algebra of intertwining kernels*. (While the whole algebra  $F_2 \otimes F_1^{\text{op}}$  is called *algebra of kernels*.)

If there exists an invariant integral  $\nu : F_1 \rightarrow \mathbf{C}$ , then the map  $i : F_1 \rightarrow F_1^*$  given by

$$(i f_1)(f_2) = \int f_1 f_2 d\nu$$

is a morphism of  $A$ -modules. This allows to assign a morphism  $F_1 \rightarrow F_2$  to any intertwining kernel  $K$ . In other words, intertwining kernels correspond to the integral operators

$$f_1 \mapsto (\text{id} \otimes \nu)(K(1 \otimes f_1))$$

which intertwine the representations of  $A$  in  $F_1$  and  $F_2$  (the functions under the symbol of integral are being multiplied in the algebra  $F_2 \otimes F_1$ , not in  $F_2 \otimes F_1^{\text{op}}$ ).

Note that the replacement of  $F_1$  by  $F_1^{\text{op}}$  results in the change of sign in the commutators, and in the limit as  $q \rightarrow 1$  it yields the above-mentioned change of sign in the formula for the Poisson bracket on the Cartesian product of the corresponding Poisson homogeneous spaces.

Consider the case  $\mathfrak{g} = \mathfrak{sl}(n+1)$ . The algebra  $U_q \mathfrak{sl}(n+1)$  can be equipped with a anti-linear anti-involution given by

$$(K_i^\pm)^* = K_i^\mp, \quad (X_i^\pm)^* = (-1)^{\delta_{i,1}} X_i^\mp$$

This is an algebra anti-automorphism and a coalgebra automorphism. Moreover, the map  $a \mapsto (S(a))^*$  is an involution. That is, the pair  $(U_q \mathfrak{sl}(n+1), *)$  denoted by  $U_q \mathfrak{su}(n, 1)$  is a Hopf  $*$ -algebra. The notation  $U_q \mathfrak{su}(n, 1)$  is justified by the fact that in the limit case  $q = 1$  the  $*$ -representations of this algebra are related to the unitary representations of  $SU(n, 1)$ .

**Definition 2.8.** Let  $A$  be a Hopf  $*$ -algebra. Suppose that a  $*$ -algebra  $F$  is equipped with an  $A$ -module structure so that the multiplication map  $m : F \otimes F \rightarrow F$  is a morphism of  $A$ -modules (that is,  $F$  is an  $A$ -module algebra).

We say that  $F$  is an  *$A$ -module  $*$ -algebra* if the following condition is satisfied:

$$(af)^* = (S(a))^* f^* \tag{2.12}$$

for any  $a \in A$  and  $f \in F$ .

Given an  $A$ -module  $*$ -algebra  $F$ , we will choose an invariant integral  $\nu : F \rightarrow \mathbb{C}$  so that

$$\int f^* d\nu = \overline{\int f d\nu}, \quad \int f^* f d\nu > 0 \text{ if } f \neq 0$$

Then we introduce a scalar product in  $F$  by letting

$$(f_1, f_2) = \int f_2^* f_1 d\nu$$

It is easy to see that (2.12) is equivalent to the condition that the quasi-regular representation of  $A$  in  $F$  given by  $\mathcal{R}(a)f = af$  is a  $*$ -representation. Indeed,  $(af_1, f_2) = \int (S^{-1}(a)f_2^*) f_1 d\nu = \int (a^* f_2)^* f_1 d\nu = (f_1, a^* f_2)$ .

Suppose that we have two  $A$ -module  $*$ -algebras  $F_1$  and  $F_2$ . It is easy to show that there exists a unique anti-linear anti-involution in the algebra of kernels such that

$$\int K^*(1 \otimes f) d\nu = \left( \int K(1 \otimes f^*) d\nu \right)^*$$

for any  $a \in A$ ,  $f \in F_1$ , and  $K \in F_2 \otimes F_1^*$ . In other words, real kernels  $K = K^*$  correspond to real integral operators.

If  $K$  is an intertwining kernel, then for any  $a \in A$  and  $f \in F_1$ , we get that

$$\begin{aligned} \int K^*(1 \otimes af) d\nu &= \left( \int K(1 \otimes (af)^*) d\nu \right)^* = \\ &= \left( \int K(1 \otimes (S(a))^* f^*) d\nu \right)^* = \left( (S(a))^* \int K(1 \otimes f^*) d\nu \right)^* = \\ &= a \left( \int K(1 \otimes f^*) d\nu \right)^* = a \int K^*(1 \otimes f) d\nu \end{aligned}$$

Therefore,  $K^*$  is also an intertwining kernel.

Given a kernel  $K$ , one can find  $K^*$ , using explicit formulas for the invariant integral. If the algebra  $F_1$  has an exact irreducible  $*$ -representation  $\pi$ , usually there is a non-negative element  $Q \in F_1$  such that

$$\int f d\nu = \text{tr } \pi(fQ)$$

for any  $f \in F_1$  (see (3.7)). In the case of a reducible representation  $\pi$  the usual trace is replaced by the trace on the corresponding von Neumann algebra. We have that

$$K^* = (1 \otimes Q)^{-1} K^{**} (1 \otimes Q) \quad (2.13)$$

In all the examples considered below in the paper the elements  $Q$  and  $Q^{-1}$  does not belong in fact to the algebra  $F_1$  of basic functions but to a space of distributions (see Section 3).

Following [3], we describe a  $U_q \mathfrak{su}(n, 1)$ -module  $*$ -algebra  $\mathbf{C}_{q, q^{-1}}^{n+1}$  which is a quantum analogue of the algebra of polynomial functions on a vector space<sup>2</sup>. As a  $*$ -algebra with unit element,  $\mathbf{C}_{q, q^{-1}}^{n+1}$  is generated by  $\{z_j\}_{j=0}^n$  and the relations

$$z_i z_j = q z_j z_i \quad (i < j), \quad z_i z_j^* = q z_j^* z_i \quad (i \neq j) \quad (2.14)$$

$$z_i z_i^* - z_i^* z_i = (-1)^{\delta_{i,0}} (q^{-2} - 1) \sum_{k>i} z_k z_k^* \quad (2.15)$$

Define an action of  $U_q \mathfrak{su}(n, 1)$  on the generators  $z_j$  by

$$X_i^+ z_j = \delta_{ij} z_{j-1} \quad (j \neq 0), \quad X_i^+ z_0 = 0 \quad (2.16)$$

$$X_i^- z_j = \delta_{i,j+1} z_{j+1} \quad (j \neq n), \quad X_i^- z_n = 0 \quad (2.17)$$

---

<sup>2</sup>In [3] this algebra was denoted by  $\mathbf{C}_{q, q^{-1}}^{n+1}$ .

$$K_i^\pm z_j = \left( \delta_{i,j+1} q^{\pm \frac{1}{2}} + \delta_{ij} q^{\mp \frac{1}{2}} \right) z_j \quad (2.18)$$

This action can be extended to an action on the whole  $\mathbf{C}_{q,q^{-1}}^{n+1}$  by (2.7), (2.8, and (2.12). For instance, we get that

$$X_i^+ z_j^* = -q^{-1} \delta_{i,j+1} z_{j+1}^* \quad (0 < j < n) \quad (2.19)$$

$$X_i^- z_j^* = q \delta_{ij} z_{j-1}^* \quad (j > 1) \quad (2.20)$$

$$K_i^\pm z_j^* = \left( \delta_{i,j+1} q^{\mp \frac{1}{2}} + \delta_{ij} q^{\pm \frac{1}{2}} \right) z_j^* \quad (2.21)$$

We will denote the generators of the algebra  $\mathbf{C}_{q,q^{-1}}^{n+1} \otimes \mathbf{C}_{q,q^{-1}}^{n+1}$  by  $z_j = z_j \otimes 1$  and  $\zeta_j = 1 \otimes z_j$ . As vector spaces, the algebras  $\mathbf{C}_{q,q^{-1}}^{n+1} \otimes \mathbf{C}_{q,q^{-1}}^{n+1}$  and  $\mathbf{C}_{q,q^{-1}}^{n+1} \otimes (\mathbf{C}_{q,q^{-1}}^{n+1})^{\text{op}}$  coincide.

Consider the kernels  $t, \tau, K_1, K_2 \in \mathbf{C}_{q,q^{-1}}^{n+1} \otimes (\mathbf{C}_{q,q^{-1}}^{n+1})^{\text{op}}$  given by

$$t = z_0 z_0^* - \sum_{j=1}^n z_j z_j^*, \quad \tau = \zeta_0 \zeta_0^* - \sum_{j=1}^n \zeta_j \zeta_j^*,$$

$$K_1 = z_0 \zeta_0^* - \sum_{j=1}^n z_j \zeta_j^*, \quad K_2 = z_0^* \zeta_0 - \sum_{j=1}^n z_j^* \zeta_j$$

**Proposition 2.9.** *The kernels  $t, \tau, K_1, K_2$  are intertwining. Moreover,  $t$  and  $\tau$  belong to the center of  $\mathbf{C}_{q,q^{-1}}^{n+1} \otimes (\mathbf{C}_{q,q^{-1}}^{n+1})^{\text{op}}$  and*

$$K_1 K_2 - q^2 K_2 K_1 - (1 - q^2) t \tau = 0 \quad (2.22)$$

**Corollary 2.10.**  *$(K_1 K_2)^m$  is an intertwining kernel for any  $m \in \mathbf{Z}_+$ .*

In the theory of representations of  $SU(n, 1)$  an important role is played by non-integer powers of the Poisson kernel  $P = K_1 K_2$ . Their quantum analogues will be defined in Section 4.

### 3 Distributions and Intertwining Kernels

Recall (cf. [6]) the definitions of the algebras of regular functions on a quantum cone  $\mathbf{C}[C^{2n+1}]_q$  and a quantum hyperboloid  $\mathbf{C}[H_+^{2n+1}]_q$ . These are the quotients of  $\mathbf{C}_{q,q^{-1}}^{n+1}$  over the ideals generated by the central elements

$$f_0 = z_0 z_0^* - \sum_{k=1}^n z_k z_k^* \quad \text{and} \quad f_1 = f_0 - 1 \quad \text{respectively}$$

Since the elements  $f_0$  and  $f_1$  are invariants and  $f_0 = f_0^*$ ,  $f_1 = f_1^*$ , the algebras  $\mathbf{C}[C^{2n+1}]_q$  and  $\mathbf{C}[H_+^{2n+1}]_q$  inherit a  $U_q\mathfrak{su}(n, 1)$ -module  $*$ -algebra structure.

However, the algebra  $\mathbf{C}[H_+^{2n+1}]_q \otimes \mathbf{C}[H_+^{2n+1}]_q^{\text{op}}$  contains only polynomial intertwining kernels. In order to define a space of distributions we first consider some corollaries of the commutation relations (2.14),(2.15).

**Proposition 3.1.** *Let  $x_i \in \mathbf{C}_{q, q^{-1}}^{n+1}$  be given by*

$$x_0 = z_0 z_0^* - \sum_{k>0} z_k z_k^*, \quad x_j = \sum_{k \geq j} z_k z_k^* \quad (j > 0) \quad (3.1)$$

*Then we have that*

$$\begin{aligned} x_i x_j &= x_j x_i \\ z_i x_j &= x_j z_i \quad (i \geq j) \\ z_i x_j &= q^2 x_j z_i \quad (i < j) \end{aligned}$$

**Corollary 3.2.** *If  $f$  is an element of either  $\mathbf{C}[C^{2n+1}]_q$  or  $\mathbf{C}[H_+^{2n+1}]_q$ , then there exists a unique decomposition*

$$f = \sum_{I, J=0} (z^*)^I f_{IJ}(x_1, \dots, x_n) z^J \quad (3.2)$$

where  $I = (i_0, \dots, i_n)$ ,  $J = (j_0, \dots, j_n)$ ,  $I \cdot J = (i_0 j_0, \dots, i_n j_n)$ , and  $(z^*)^I = (z_0^*)^{i_0} (z_1^*)^{i_1} \dots (z_n^*)^{i_n}$ ,  $z^J = z_0^{j_0} z_1^{j_1} \dots z_n^{j_n}$ .

Of course, here we have preserved the same notation for the generators of the quotient algebras, the sum in (3.2) is finite, and the coefficients  $f_{IJ}$  are polynomials.

The action of the generators  $X_i^\pm$  and  $K_i^\pm$  of  $U_q\mathfrak{su}(n, 1)$  on the terms of the sum in (3.2) is described by  $q$ -difference operators, as follows from (2.7), (2.8), and the following simple statement.

**Proposition 3.3.** *For any polynomial  $f(x_1, \dots, x_n)$  in the algebra  $\mathbf{C}_{q, q^{-1}}^{n+1}$ , we have that*

$$X_i^+ f = q^{\frac{3}{2}} z_i^* (B_i f)(x_1, \dots, x_i, q^2 x_{i+1}, \dots, q^2 x_n) z_{i-1} \quad (3.3)$$

$$X_i^- f = -(-1)^{\delta_{i,1}} q^{\frac{3}{2}} z_{i-1}^* (B_i f)(x_1, \dots, x_i, q^2 x_{i+1}, \dots, q^2 x_n) z_i \quad (3.4)$$

$$K_i^\pm f = f \quad (3.5)$$

where

$$\begin{aligned} B : f(t) &\mapsto \frac{f(q^2 t) - f(t)}{q^2 t - t}, \\ B_i &= \underbrace{\text{id} \otimes \dots \otimes \text{id}}_{i-1} \otimes B \otimes \text{id} \otimes \dots \otimes \text{id} \end{aligned}$$

The equalities (3.3)-(3.5) and the commutation relations

$$\begin{aligned} z_i f(x_1, \dots, x_n) &= f(x_1, \dots, x_i, q^2 x_{i+1}, \dots, q^2 x_n) z_i \\ z_i^* f(x_1, \dots, x_n) &= f(x_1, \dots, x_i, q^{-2} x_{i+1}, \dots, q^{-2} x_n) z_i^* \end{aligned}$$

enable us to extend the ring of polynomials  $\mathbf{C}[x_1, \dots, x_n]$  whose elements have been so far the coefficients  $f_{IJ}$  of the sum in (3.2). Namely, let

$$\mathfrak{M}_\beta \stackrel{\text{def}}{=}$$

$$\left\{ \left( q^{2(m_1+\beta)}, \dots, q^{2(m_n+\beta)} \right) \in \mathbf{R}^n \mid (m_1, \dots, m_n) \in \mathbf{Z}^n, m_1 < m_2 < \dots < m_n \right\}$$

where  $0 \leq \beta < 1$ .

Consider the algebra  $\mathcal{D}_\beta$  of functions on  $\mathbf{R}^n$  with finite support ( $\#(\text{supp } f) < \infty$ ) such that  $\text{supp } f \subset \mathfrak{M}_\beta$ . We call *basic functions* on the quantum hyperboloid sums of the form (3.2) where  $f_{IJ} \in \mathcal{D}_\beta$ . Similarly we define basic functions on the quantum cone. The spaces of basic functions will be denoted by  $\mathcal{D}(H_+^{2n+1})_{q,\beta}$  and  $\mathcal{D}(C^{2n+1})_{q,\beta}$  respectively. It is clear that they can be equipped with a  $U_q \mathfrak{su}(n, 1)$ -module  $*$ -algebra structure by (2.16)-(2.21) and (3.3)-(3.5).

Let  $\{f^{(m)}\}$  be a sequence of elements of  $\mathbf{C}[H_+^{2n+1}]_q$  and  $f \in \mathcal{D}(H_+^{2n+1})_{q,\beta}$ . We will say that the sequence  $f^{(m)}$  converges to  $f$  if for any  $I, J$  we have pointwise on  $\mathfrak{M}_\beta$  that

$$\lim_{m \rightarrow \infty} f_{IJ}^{(m)} = f_{IJ} \quad \text{and} \quad \# \left( \bigcup_m \left\{ (I, J) \mid f_{IJ}^{(m)} \neq 0 \right\} \right) < \infty$$

The same way we define convergence of a sequence of elements of  $\mathbf{C}[C^{2n+1}]_q$  to an element of  $\mathcal{D}(C^{2n+1})_{q,\beta}$ .

**Proposition 3.4.** (i) *The multiplication and the action of  $U_q \mathfrak{su}(n, 1)$  can be extended continuously from  $\mathbf{C}[C^{2n+1}]_q$  to  $\mathcal{D}(C^{2n+1})_{q,\beta}$  and from  $\mathbf{C}[H_+^{2n+1}]_q$  to  $\mathcal{D}(H_+^{2n+1})_{q,\beta}$ .*

(ii) *The formula*

$$\int f d\nu_\beta = (1 - q^2)^n \sum_{(x_1, \dots, x_n) \in \mathfrak{M}_\beta} f_{00}(x_1, \dots, x_n) x_1 \dots x_n \quad (3.6)$$

*defines an invariant integral on both  $\mathcal{D}(C^{2n+1})_{q,\beta}$  and  $\mathcal{D}(H_+^{2n+1})_{q,\beta}$ .*

(iii) *The involution  $*$  can be extended continuously to  $\mathcal{D}(C^{2n+1})_{q,\beta}$  and  $\mathcal{D}(H_+^{2n+1})_{q,\beta}$ . Then we have that  $\int f^* f d\nu_\beta > 0$  for any  $f \neq 0$ .*

*Proof.* When checking the statements (i) and (ii) the Hopf algebra  $U_q \mathfrak{sl}(n+1)$  can be replaced by a Hopf subalgebra isomorphic to  $U_q \mathfrak{sl}(2)$  and generated by the elements  $X_i^\pm$  and  $K_i^\pm$ , with  $i$  being fixed. Then it suffices to use (2.16)-(2.21) and (3.3)-(3.5).

Let us prove (iii). The involution can be extended in an obvious way. The positivity of the integral (3.6) follows from the fact that

$$\int f d\nu = (1 - q^2)^n \frac{1}{2\pi} \int \text{tr } \pi_\varepsilon^{(\beta, \varphi)}(f x_1 \dots x_n) \quad (3.7)$$

where in the case  $\varepsilon = 0$   $\pi_0^{(\beta, \varphi)}$  is a certain  $*$ -representation of  $\mathcal{D}(C^{2n+1})_{q, \beta}$  and in the case  $\varepsilon = 1$   $\pi_1^{(\beta, \varphi)}$  is a certain  $*$ -representation of  $\mathcal{D}(H_+^{2n+1})_{q, \beta}$ .

The representation  $\pi_\varepsilon^{(\beta, \varphi)}$  is defined in the space of functions  $\psi$  with finite support on  $\mathbf{Z} \times \mathbf{Z}_+^{n-1}$ . The scalar product in this space is given in a standard way by

$$(\psi_1, \psi_2) = \sum_{(j_1, \dots, j_n)} \psi_1(j_1, \dots, j_n) \overline{\psi_2(j_1, \dots, j_n)}$$

The operators  $\pi_\varepsilon^{(\beta, \varphi)}(z_k)$  and  $\pi_\varepsilon^{(\beta, \varphi)}(f(x_1, \dots, x_n))$  are given by

$$\left( \pi_\varepsilon^{(\beta, \varphi)}(z_n) \psi \right) (j_1, \dots, j_n) = e^{-i\varphi} q^{\beta + j_1 + \dots + j_n} \psi(j_1, \dots, j_n) \quad (3.8)$$

$$\begin{aligned} \left( \pi_\varepsilon^{(\beta, \varphi)}(z_k) \psi \right) (j_1, \dots, j_n) &= e^{-i\varphi} q^{\beta + j_1 + \dots + j_k} (1 - q^{2j_{k+1}}) \cdot \\ &\cdot \psi(j_1, \dots, j_k, j_{k+1} + 1, j_{k+2}, \dots, j_n), \quad (0 < k < n) \end{aligned} \quad (3.9)$$

$$\left( \pi_\varepsilon^{(\beta, \varphi)}(z_0) \psi \right) (j_1, \dots, j_n) = q^\beta \left( \varepsilon + q^{2(j_1 + \beta)} \right) \psi(j_1 + 1, j_2, \dots, j_n) \quad (3.10)$$

$$\left( \pi_\varepsilon^{(\beta, \varphi)}(f) \psi \right) (j_1, \dots, j_n) = f \left( q^{2(\beta + j_1)}, \dots, q^{2(\beta + j_1 + \dots + j_n)} \right) \psi(j_1, \dots, j_n) \quad (3.11)$$

Then the positivity of the integral follows from the exactness of the representations  $\oplus \int \pi_0^{(\beta, \varphi)} d\varphi, \oplus \int \pi_1^{(\beta, \varphi)} d\varphi$ .

*Remarks.* (i) In what follows the algebras  $\mathcal{D}(C^{2n+1})_{q, \beta}$  and  $\mathcal{D}(H_+^{2n+1})_{q, \beta}$  will play the roles of the algebras  $F_1$  and  $F_2$  (see Section 2) respectively.

(ii) It is easy to get the full list of irreducible  $*$ -representations  $\pi$  of the algebras  $\mathcal{D}(C^{2n+1})_{q, \beta}$  and  $\mathcal{D}(H_+^{2n+1})_{q, \beta}$  for which the operators  $\pi(x_1), \dots, \pi(x_n)$  have at least one common eigen-vector.

The principal series of representations is characterized by the condition  $\pi(x_n) \neq 0$  and are parameterized by the points  $(\beta, \varphi)$  of a two-dimensional torus. They are given by (3.8)-(3.11). The representations of degenerate series are either one-dimensional or obtained from representations for smaller dimensions  $n$ :

$$\begin{aligned} \mathbf{C} [C^{2n+1}]_q / (z_n = z_n^* = 0) &\simeq \mathbf{C} [C^{2n-1}]_q \\ \mathbf{C} [H_+^{2n+1}]_q / (z_n = z_n^* = 0) &\simeq \mathbf{C} [H_+^{2n-1}]_q \end{aligned}$$

The limit  $q \rightarrow 1$  equips  $H_+^{2n+1}$  and  $C^{2n+1} \setminus \{0\}$  with a Poisson manifold structure and establishes a correspondence between the principal series representations and the symplectic leaves of maximal dimension ( $\text{agr } z_n = \varphi$ ). The invariant integral gives the Liouville measure in the limit, and the degenerate series representations correspond to the symplectic leaves of dimension  $d < 2n$  (cf [6]). The parameter  $\beta$ , unlike  $\varphi$ , does not have a simple classical analogue ( $q \rightarrow 1$ ), as  $\beta$  distinguishes the homogeneous components of the quantum  $SU(n, 1)$ -spaces, while in the case  $q = 1$  the group  $SU(n, 1)$  acts transitively on  $H_+^{2n+1}$  and  $C^{2n+1} \setminus \{0\}$ .

In the conclusion of the section, we define the spaces of distributions and the space of the kernels of integral operators.

Equip the algebras  $\mathcal{D}(C^{2n+1})_{q,\beta}$  and  $\mathcal{D}(H_+^{2n+1})_{q,\beta}$  with the weakest topology in which are continuous the linear functionals  $f \mapsto f_{IJ}(x_1, \dots, x_n)$  for any  $I, J$  and  $x_1, \dots, x_n \in \mathfrak{M}_\beta$ .

The completions  $\mathcal{D}(C^{2n+1})'_{q,\beta}$  and  $\mathcal{D}(H_+^{2n+1})'_{q,\beta}$  will be called *the spaces of distributions*. By continuity, they have a  $U_q \mathfrak{sl}(n+1)$ -module structures. As follows from (2.9), the pairings

$$\begin{aligned} \mathcal{D}(C^{2n+1})'_{q,\beta} \otimes \mathcal{D}(C^{2n+1})_{q,\beta} &\rightarrow \mathbf{C} \\ \mathcal{D}(H_+^{2n+1})'_{q,\beta} \otimes \mathcal{D}(H_+^{2n+1})_{q,\beta} &\rightarrow \mathbf{C} \\ f \otimes \psi &\mapsto \int f \psi d\nu \end{aligned}$$

yield canonical isomorphisms of the  $U_q \mathfrak{sl}(n+1)$ -modules of distributions and the modules dual to the modules of basic functions.

By continuity we define the product of a basic function and a distribution, which allows to equip  $\mathcal{D}(H_+^{2n+1})'_{q,\beta}$  with a structure of  $\mathcal{D}(H_+^{2n+1})_{q,\beta}$ -bimodule and  $\mathcal{D}(C^{2n+1})'_{q,\beta}$  with a structure of  $\mathcal{D}(C^{2n+1})_{q,\beta}$ -bimodule. This structure is compatible with the structure of  $U_q \mathfrak{sl}(n+1)$ -module in the sense of (2.7), (2.8).

The distributions are decomposable in the series (3.2), and they can be identified with formal series of the form (3.2), with the coefficients  $f_{IJ}$  being functions on  $\mathfrak{M}_\beta$ . The topology coincides with the topology of the pointwise convergence of the coefficients  $f_{IJ}(x_1, \dots, x_n)$ .

For the tensor product  $\mathbf{C}[H_+^{2n+1}]_q \otimes \mathbf{C}[C^{2n+1}]_q^{\text{op}}$  the decomposition (3.2) is of the form

$$f = \sum_{I \cdot J=0} \sum_{I' \cdot J'=0} (z^*)^I \zeta^{I'} f_{IJ I' J'}(x_0, \dots, x_n; \xi_0, \dots, \xi_n) z^J (\zeta^*)^{J'} \quad (3.12)$$

where  $\zeta^{I'} = \zeta_0^{i'_0} \dots \zeta_n^{i'_n}$ ,  $(\zeta^*)^{J'} = (\zeta_0^*)^{j'_0} \dots (\zeta_n^*)^{j'_n}$ , and

$$\xi_0 = \zeta_0^* \zeta_0 - \sum_{k>0} \zeta_k^* \zeta_k, \quad \xi_j = \sum_{k \geq j} \zeta_k^* \zeta_k \quad (j \neq 0)$$

Passing from the finite sums to the formal series and from the polynomials  $f_{IJJ'I'}$  to functions on  $\mathfrak{M}_\beta \times \mathfrak{M}_\beta$ , we get the space  $\mathcal{K}(H_+^{2n+1}, C^{2n+1})_{q,\beta}$  of the kernels of integral operators. In the same way as above, we equip the space  $\mathcal{K}(H_+^{2n+1}, C^{2n+1})_{q,\beta}$  with compatible  $U_q \mathfrak{sl}(n+1) \otimes U_q \mathfrak{sl}(n+1)$ -module and  $\mathcal{D}(H_+^{2n+1})_{q,\beta} \otimes \mathcal{D}(C^{2n+1})_{q,\beta}$ -bimodule structures. We will call *intertwining kernels* the solutions to the system (2.11) in the space  $\mathcal{K}(H_+^{2n+1}, C^{2n+1})_{q,\beta}$ . As was noted in Section 2, the intertwining kernels correspond to intertwining integral operators  $\mathcal{D}(C^{2n+1})_{q,\beta} \rightarrow \mathcal{D}(H_+^{2n+1})_{q,\beta}$ .

## 4 Non-Integer Powers of the Poisson Kernel

As will be shown, the power  $P^\lambda$  of the Poisson kernel for  $\lambda \in \mathbf{Z}_+$  can be decomposed into  $Q$ -binomial “series”, with coefficients being rational functions in  $q^{2\lambda}$ . This will enable us to perform the analytic continuation in the parameter  $\lambda$ . The convergence of the resulting series in the space of kernels  $\mathcal{K}(H_+^{2n+1}, C^{2n+1})_{q,\beta}$  can be established in the same way as the following proposition.

**Proposition 4.1.** *Suppose that  $K' = \sum_{j=1}^{n-1} z_j \zeta_j^*$  and  $K'' = \sum_{j=1}^{n-1} q^{-2j} z_j^* \zeta_j$ . Then the series*

$$\sum_{m=0}^{\infty} \sum_{\substack{m_1 + m_2 = m \\ m_1, m_2 \geq 0}} c_{m_1 m_2} (K'')^{m_2} (K')^{m_1} \quad (4.1)$$

*converges in  $\mathcal{K}(H_+^{2n+1}, C^{2n+1})_{q,\beta}$  for any  $c_{m_1 m_2} \in \mathbf{C}$ .*

*Proof.* Choosing a point  $(x_1, \dots, x_n; \xi_1, \dots, \xi_n) \in \mathfrak{M}_\beta \times \mathfrak{M}_\beta$  and a quadruple of multi-indices  $(I, J, I', J')$ , we bring the terms of the series (4.1) to the “normal form” (3.12) by using the commutation relations (2.14), (2.15). Then it suffices to show that for some  $M \in \mathbf{Z}_+$  the terms with the indices  $m > M$  give the zero contribution to  $f_{IJJ'I'}(x_0, \dots, x_n; \xi_0, \dots, \xi_n)$ . But this follows from the definition of  $\mathfrak{M}_\beta$  and the following identities in the algebra  $\mathbf{C}_{q, q^{-1}}^{n+1} \otimes \left( \mathbf{C}_{q, q^{-1}}^{n+1} \right)^{\text{op}}$ :

$$\begin{aligned} & (z_1^*)^{m_1} (z_2^*)^{m_2} \dots (z_{n-1}^*)^{m_{n-1}} z_1^{m_1} z_2^{m_2} \dots z_{n-1}^{m_{n-1}} = \\ & = \text{const}_1(m_1, \dots, m_{n-1}) \prod_{k=1}^{n-1} \prod_{j=1}^{m_k} (x_k - q^{-2} x_{k+1}), \\ & \zeta_1^{m_1} \zeta_2^{m_2} \dots \zeta_{n-1}^{m_{n-1}} (\zeta_1^*)^{m_1} (\zeta_2^*)^{m_2} \dots (\zeta_{n-1}^*)^{m_{n-1}} = \\ & = \text{const}_2(m_1, \dots, m_{n-1}) \prod_{k=1}^{n-1} \prod_{j=1}^{m_k} (\xi_k - q^{-2} \xi_{k+1}) \end{aligned}$$

By (2.22), for  $\lambda \in \mathbf{Z}_+$  the powers of  $P = K_1 K_2$  are given by

$$P^\lambda = q^{\lambda(\lambda+1)} (z_0^* \zeta_0 - K'' - q^{-2n} z_n^* \zeta_n)^\lambda (z_0 \zeta_0^* - K' - z_n \zeta_n^*)^\lambda \quad (4.2)$$

Note that the summands in any of the parentheses quasi-commute. For instance, we have that

$$\begin{aligned} (z_0 \zeta_0^*) K' &= q^2 K' (z_0 \zeta_0^*), & K' (z_n \zeta_n^*) &= q^2 (z_n \zeta_n^*) K' \\ (z_0^* \zeta_0) K'' &= q^{-2} K'' (z_0^* \zeta_0), & K'' (z_n^* \zeta_n) &= q^{-2} (z_n^* \zeta_n) K'' \end{aligned}$$

This allows us to use the well-known  $q$ -analogue of the Newton binomial formula:

**Proposition 4.2.** *If  $a$  and  $b$  are elements of an associative algebra such that  $ab = q^2 ba$ , then*

$$\begin{aligned} (a + b)^m &= \sum_{j=0}^m \frac{(q^2; q^2)_m}{(q^2; q^2)_j (q^2; q^2)_{m-j}} b^j a^{m-j}, \\ (t; q^2)_k &\stackrel{\text{def}}{=} (1-t)(1-q^2 t) \dots (1-q^{2(k-1)} t) \end{aligned}$$

**Corollary 4.3.** *We have that*

$$\begin{aligned} (z_0 \zeta_0^* - K' - z_n \zeta_n^*)^\lambda &= \sum_k \frac{(q^{2\lambda}; q^{-2})_k}{(q^2; q^2)_k} \sum_i \frac{(q^{2(\lambda+k)}; q^{-2})_i}{(q^2; q^2)_i} \cdot \\ &\quad \cdot q^{-2(i+k)(\lambda-i-k)} (z_0 \zeta_0^*)^{\lambda-i-k} (-z_n \zeta_n^*)^i (-K')^k, \\ (z_0^* \zeta_0 - K'' - q^{-2n} z_n^* \zeta_n)^\lambda &= \sum_k \frac{(q^{-2\lambda}; q^2)_k}{(q^{-2}; q^{-2})_k} \sum_i \frac{(q^{-2(\lambda-k)}; q^2)_i}{(q^{-2}; q^{-2})_i} \cdot \\ &\quad \cdot q^{2ki} (-K'')^k (-q^{-2n} z_n^* \zeta_n)^i (z_0^* \zeta_0)^{\lambda-i-k} \end{aligned}$$

It follows from (4.2) and Corollary 4.3 that  $P^\lambda$  can be decomposed into a sum

$$P^\lambda = \sum_{l_1, l_2} (-K'')^{l_1} f_{l_1 l_2} (q^{2\lambda}) (K')^{l_2} \quad (4.3)$$

of the form (4.1), where

$$\begin{aligned} f_{l_1 l_2} (q^{2\lambda}) &= \sum_{i_1, i_2} q^{\lambda(\lambda+1)} \text{const}(\lambda, i_1, i_2, l_1, l_2) \cdot \\ &\quad \cdot (-q^{-2n} z_n^* \zeta_n)^{i_1} (z_0^* \zeta_0)^{\lambda-l_1-i_1} (z_0 \zeta_0^*)^{\lambda-l_2-i_2} (-z_n \zeta_n^*)^{i_2} \\ \text{const}(\lambda, i_1, i_2, l_1, l_2) &= q^{l_1(l_1+1)-l_2(l_2+1)} \frac{(q^{-2\lambda}; q^2)_{l_1} (q^{-2\lambda}; q^2)_{l_2}}{(q^2; q^2)_{l_1} (q^2; q^2)_{l_2}} \cdot \\ &\quad \cdot (-1)^{l_1+l_2+i_1+i_2} q^{i_1(i_1+1)-i_2(i_2+1)+2i_1 l_1+2i_2 l_2+2(i_2+l_2)^2} \cdot \\ &\quad \cdot \frac{(q^{-2(\lambda-l_1)}; q^2)_{i_1} (q^{-2(\lambda+l_1)}; q^2)_{i_2}}{(q^2; q^2)_{i_1} (q^2; q^2)_{i_2}} \end{aligned} \quad (4.5)$$

**Proposition 4.4.** *The powers of the Poisson kernel  $P^\lambda$  have a decomposition of the form (3.12), with the coefficients of the power series  $\xi_1^{-\lambda} f_{IJI'J'}$  being polynomials in  $q^{2\lambda}$ ,  $q^{-2\lambda}$  (in particular, rational functions in  $q^{2\lambda}$ ).*

To prove it, it suffices to go to a “normal” ordering of the generators in  $P^\lambda$  (as in (3.12)) by using (4.3)-(4.5) and the following simple statement.

**Proposition 4.5.** *Suppose that  $m_1 \leq \lambda$ ,  $m_2 \leq \lambda$ ,  $m = \max(m_1, m_2)$ . Then*

$$q^{\lambda(\lambda+1)} (z_0^* \zeta_0)^{\lambda-m_1} (z_0 \zeta_0^*)^{\lambda-m_2} = q^{m(2\lambda-m+1)} (z_0^* \zeta_0)^{m-m_1} \cdot \frac{(-q^{-2(\lambda-m)} x_1; q^2)_\infty}{(-x_1; q^2)_\infty} \xi_1^{\lambda-m} (z_0 \zeta_0^*)^{m-m_2} \quad (4.6)$$

where

$$(t; q^2)_\infty = \prod_{j=0}^{\infty} (1 - q^{2j} t) = \sum_{l=1}^{\infty} \frac{t^l q^{l(l-1)}}{(q^2; q^2)_l}$$

*Remark.* The crucial point in the proof of Proposition 4.5 is the elimination of the multiple  $q^{\lambda^2}$  due to the fact that  $\zeta_0^{\lambda-m} (\zeta_0^*)^{\lambda-m} = q^{-(\lambda+m)(\lambda+m+1)} \xi_1^{\lambda-m}$ .

Now, by analytical continuation, we can define  $P^\lambda$  as a power series for any  $\lambda \in \mathbf{R}$ . The formulas (4.3)-(4.6) hold in this case. Let us bring the coefficients  $f_{l_1 l_2}$  of the series (4.3) to the form (3.12):

$$f_{l_1 l_2} = \sum_{\substack{j_0, j_n, k_0, k_n \\ j_0 k_0 = j_n k_n = 0}} (z_0^* \zeta_0)^{j_0} (-q^{-2n} z_n^* \zeta_n)^{j_n} \xi_1^\lambda \cdot f_{l_1 l_2 j_0 j_n k_0 k_n} (q^{2\lambda}; x_1, x_n, \xi_1^{-1} \xi_n) (z_0 \zeta_0^*)^{k_0} (z_n \zeta_n^*)^{k_n} \quad (4.7)$$

As can be seen from the proof of Proposition 4.1, in order to get a convergent series in the space of kernels  $\mathcal{K}(H_+^{2n+1}, C^{2n+1})_{q, \beta}$  it suffices to establish convergence of any of the power series  $f_{l_1 l_2 j_0 j_n k_0 k_n}$  in some neighbourhood of zero and to continue them analytically.

Recall the definition of the *basic hypergeometric series*  ${}_{r+1}\Phi_{r+j}$  (cf. [1]):

$${}_{r+1}\Phi_{r+j} \left( \begin{matrix} a_0, \dots, a_r \\ b_1, \dots, b_{r+j} \end{matrix}; q, x \right) \stackrel{\text{def}}{=} \sum_{m=0}^{\infty} \frac{(a_0; q)_m \dots (a_r; q)_m (-1)^{jm} q^{ij(m-1)/2}}{(b_1; q)_m \dots (b_{r+j}; q)_m (q; q)_m} x^m$$

**Proposition 4.6.** *we have that*

$$f_{000000} (q^{2\lambda}; x_1, x_n, \xi_1^{-1} \xi_n) = \frac{(-q^{-2\lambda} x_1; q^2)_\infty}{(-x_1; q^2)_\infty} {}_2\Phi_2 \left( \begin{matrix} q^{-2\lambda}, & q^{-2\lambda} \\ q^2, & -q^{-2\lambda} x_1 \end{matrix}; q^2, -q^{2(\lambda+n-2)} x_n \xi_1^{-1} \xi_n \right) \quad (4.8)$$

The proof is straightforward and based on (4.4), (4.5) and Corollary 4.5.

**Proposition 4.7.** *For any  $\lambda \in \mathbf{R}$ , the series  $f_{l_1 l_2 j_0 j_n k_0 k_n}$  converges in some neighbourhood of zero and can be continued analytically in the region  $x_1 > 0$ .*

In the special case  $l_1 = l_2 = j_0 = j_n = k_0 = k_n = 0$  this proposition follows from the previous one. In the general case it can be proved in the way analogous to that of the proof of Proposition 4.6. Namely, the analytic continuation can be achieved by applying (4.6), while the convergence is provided by the multiple  $q^{i_1^2 + i_2^2}$  in (4.5).

**Definition 4.8.** For any real  $\lambda$  the formulas (4.3), (4.7) and Proposition 4.7 determine a convergent series in the space of kernels  $\mathcal{K}(H_+^{2n+1}, C^{2n+1})_{q, \beta}$ . We call its sum *the  $\lambda$ -th power of the Poisson kernel* and denote it by  $P^\lambda$ .

Following (2.13), we define an anti-linear anti-involution in the space of kernels by

$$\begin{aligned} z_j &\mapsto z_j^*, & f(x_1, \dots, x_n; \xi_1, \dots, \xi_n) &\mapsto f(x_1, \dots, x_n; \xi_1, \dots, \xi_n), \\ \zeta_j &\mapsto (\xi_1 \dots \xi_n)^{-1} \zeta_j^* (\xi_1 \dots \xi_n) = q^{2(j-n)} \xi_j^* \end{aligned}$$

It is clear that  $(K'')^* = q^{-2n} K'$ . Therefore,  $P^\lambda = (P^\lambda)^*$ .

**Proposition 4.9.** *For any  $\lambda \in \mathbf{R}$ , the kernel  $P^\lambda \in \mathcal{K}(H_+^{2n+1}, C^{2n+1})_{q, \beta}$  is intertwining.*

*Proof.* Consider the kernel  $L(\lambda) = (\xi \otimes 1 - 1 \otimes S^{-1}(\xi)) (P^\lambda)$ . By (2.14), (2.15), (3.3)-(3.5), (2.16)-(2.21), the coefficients in the decomposition of  $L(\lambda)$  in a series of the form (3.12) are analytical functions in  $x_1, \dots, x_n, \xi_1^{-1} \xi_2, \dots, \xi_1^{-1} \xi_n$  in the region  $x_1 > 0$ . Close to zero they can be decomposed into power series, with coefficients being rational functions in  $q^{2\lambda}$ . But at the points  $q^2, q^4, q^6, \dots$  these rational functions have zeroes, since the kernel  $P$  and hence the kernels  $P^\lambda$  ( $\lambda \in \mathbf{Z}_+$ ) are intertwining. Therefore,  $L(\lambda) \equiv 0$ . The proposition is proved.

Analogously, one can prove that  $P^\lambda P^m = P^m P^\lambda = P^{\lambda+m}$  for any  $\lambda \in \mathbf{R}$ ,  $m \in \mathbf{Z}_+$ .

Some applications of the intertwining kernels are related to the fact that they are generating functions for spherical functions on the quantum homogeneous spaces.

*Example.* When  $n = 1$ ,  $\lambda = -l - 1$ ,  $l \in \mathbf{Z}_+$ , the function  $f_{000000}$  is a zonal spherical function corresponding to the spin  $l$  finite-dimensional representation

of  $U_q \mathfrak{sl}(2)$ . By (4.8), we get that

$$\begin{aligned} f_{0000000} &= \frac{(-q^{2(l+1)}x_1; q^2)_\infty}{(-x_1; q^2)_\infty} {}_2\Phi_2 \left( \begin{matrix} q^{2(l+1)}, & q^{2(l+1)} \\ q^2, & q^{2(l+1)}x_1 \end{matrix} ; q^2, -q^{-2l}x_1 \right) = \\ &= {}_2\Phi_1 \left( \begin{matrix} q^{2(l+1)}, q^{-2l} \\ q^2 \end{matrix} ; q^2, -qx \right) \end{aligned} \quad (4.9)$$

Here we have used the  $q$ -analogue of the Pfaff transform (cf. (1.32) in [1]). The right hand side of (4.9) is the zonal spherical function (cf. [7]). The generating function for the  $q$ -analogues of the Clebsch-Gordan coefficients obtained in [8] also is an intertwining kernel in the sense of Section 2 of the present paper.

## 5 Quantum Radon Transform

We are interested in functions of the Poisson kernel which are not necessarily polynomials. In Section 4 we defined the powers  $P^\lambda \in \mathcal{K}(H_+^{2n+1}, C^{2n+1})_{q,\beta}$  for any  $\lambda \in \mathbf{R}$ . In the present section we will obtain their decomposition of the form

$$P^\lambda = \sum_{j \in \mathbf{Z}} q^{2(j+\beta)\lambda} R_j \quad (5.1)$$

where  $R_j \in \mathcal{K}(H_+^{2n+1}, C^{2n+1})_{q,\beta}$ . The formula (5.1) allows to interpret the kernels  $R_j$  as  $\delta$ -functions  $\delta(P - q^{2(j+\beta)})$  and the integral transforms with these kernels as quantum analogues of the Radon transform.

In Section 4 we were interested in the functions  $f_{l_1 l_2 j_0 j_n k_0 k_n}$  for a fixed  $\lambda$ . Now we fix  $(x_1, \dots, x_n) \in \mathfrak{M}_\beta$ ,  $(\xi_1, \dots, \xi_n) \in \mathfrak{M}_\beta$ .

**Proposition 5.1.** *After the change of the variable  $q^{2\lambda} = u$  the function  $f_{l_1 l_2 j_0 j_n k_0 k_n}(u)$  is holomorphic in some neighbourhood of zero but the point  $u = 0$  for any  $l_1, l_2, j_0, j_n, k_0, k_n$ .*

This statement follows from (4.5), (4.6) and the commutation relations between the generators of the algebra  $\mathbf{C}_{q,q^{-1}}^{n+1} \otimes (\mathbf{C}_{q,q^{-1}}^{n+1})^{\text{op}}$ .

**Corollary 5.2.** *There exist such functions  $r_{l_1 l_2 j_0 j_n k_0 k_n}^{(m)}$  on  $\mathfrak{M}_\beta \times \mathfrak{M}_\beta$ , that*

$$f_{l_1 l_2 j_0 j_n k_0 k_n} = \sum_{m \in \mathbf{Z}} r_{l_1 l_2 j_0 j_n k_0 k_n}^{(m)} q^{2m\lambda} \quad (5.2)$$

Plugging (5.2) into (4.7), we get that the dependence on  $\lambda$  is reflected in the factor  $(\xi_1 q^{2m})^\lambda$  only. Therefore, only those values of  $\xi_1$  make a contribution into  $\delta(P - a)$  for which  $\xi_1 = a q^{-2m}$ .

**Definition 5.3.** The kernel  $\delta(P - q^{2\beta})$  of the quantum Radon transform is the sum of the series (4.3) where

$$f_{l_1 l_2} = \sum_{j_0 k_0 = j_n k_n = 0} (z_0^* \zeta_0)^{j_0} (-q^{-2n} z_n^* \zeta_n)^{j_n} \cdot r_{l_1 l_2 j_0 j_n k_0 k_n}^{(m)} \Big|_{m=\frac{1}{2} \log_q \xi - \beta} (z_0^* \zeta_0)^{k_0} (z_n^* \zeta_n)^{k_n}$$

The convergence of the series (4.3) in the space  $\mathcal{K}(H_+^{2n+1}, C^{2n+1})_{q, \beta}$  can be proved in the same way as Proposition 4.1. Similarly to the above definition, one can introduce the kernels  $R_j = \delta(P - q^{2(j+\beta)})$  for any integer  $j$ . The decomposition (5.1) follows from the above constructions.

**Proposition 5.4.** The kernel  $\delta(P - q^{2\beta})$  is intertwining.

This follows from the formula

$$r_{l_1 l_2 j_0 j_n k_0 k_n}^{(m)} = \frac{1}{2\pi i} \int_{|u|=\varepsilon} \frac{1}{u^{m+1}} f_{l_1 l_2 j_0 j_n k_0 k_n} du$$

and from the fact that the action of  $U_q \mathfrak{sl}(n+1) \otimes U_1 \mathfrak{sl}(n+1)$  commutes with the integration. Indeed, repeating for small  $|u| = q^{\operatorname{Re} \lambda}$  the proof of Proposition 4.9, we get that the functions under the integration symbol – which are obtained by applying the operator  $\xi \otimes 1 - 1 \otimes S^{-1}(\xi)$  to the kernel  $\delta(P - q^{2\beta})$  – are equal to zero.

*Remark.* Suppose that  $m \in \mathbf{Z}_+$ . Then  $P^\lambda P^m = P^m P^\lambda = P^{\lambda+m}$  implies that  $P^m \delta(P - q^{2\beta}) = \delta(P - q^{2\beta}) P^m = q^{2\beta m} P^m$ .

## 6 Intertwining Kernels and Hypergeometric Functions

Recall that the algebra of intertwining kernels is a subalgebra of  $F_2 \otimes F_1^{\operatorname{op}}$  and is defined by (2.11). In what follows the role of  $F_1$  is played by  $\mathbf{C}[C^{2n+1}]_q$  (cf. Section 3), and the role of  $F_2$  by the algebra  $(\oplus_{i=1}^N \mathbf{C}^{n+1})_{q, q^{-1}}^{\operatorname{op}}$  defined below.

We will use the  $R$ -matrix notation of [3]. Let  $e_{ij} \in \operatorname{Mat}_{n+1}(\mathbf{C})$  be the matrix unit and

$$\begin{aligned} R' &= q^2 \sum_i e_{ii} \otimes e_{ii} + q \sum_{i \neq j} e_{ij} \otimes e_{ji} + (q^2 - 1) \sum_{i < j} e_{ii} \otimes e_{jj}, \\ R'' &= q^{-1} \sum_i e_{ii} \otimes e_{ii} + q \sum_{i \neq j} e_{ij} \otimes e_{ji} + (q - q^{-1}) \sum_{i < j} q^{i-j} e_{ij} \otimes e_{ij}. \end{aligned}$$

Define  $(\oplus_{i=1}^N \mathbf{C}^{n+1})_{q,q^{-1}}^{\text{op}}$  as the  $*$ -algebra generated by  $z_{ij}$  ( $1 \leq i \leq N$ ,  $0 \leq j \leq n$ ) and the relations

$$\begin{aligned} \sum_{i_1 j_1} R'_{ij i_1 j_1} z_{b i_1} z_{a j_1} &= q z_{a i} z_{b j} \quad (a < b), \\ \sum_{i_1 j_1} R''_{ij i_1 j_1} (\varepsilon_{i_1} z_{b i_1}^*) z_{a j_1} &= q^{-1} z_{a i} (\varepsilon_j z_{b j}^*), \end{aligned}$$

where  $\varepsilon_i = (-1)^{\delta_{i0}} q^{-i}$ .

Define an action of the generators  $X_i^\pm, K_i^\pm$  of  $U_q \mathfrak{su}(n, 1)$  on the generators  $z_{ij}$  of  $(\oplus_{i=1}^N \mathbf{C}^{n+1})_{q,q^{-1}}^{\text{op}}$  by (2.16)-(2.18), replacing  $z_j, z_{j-1}, z_{j+1}$  by  $z_{ij}, z_{i,j-1}, z_{i,j+1}$  respectively. Extend this action so that  $(\oplus_{i=1}^N \mathbf{C}^{n+1})_{q,q^{-1}}^{\text{op}}$  becomes a  $U_q \mathfrak{su}(n, 1)$ -module  $*$ -algebra.

The uniqueness of such an extension is obvious due to the conditions (2.7), (2.8), (2.12). The existence follows from the fact that the matrices  $R'$  and  $R''$  correspond to linear operators which intertwine certain representations of  $U_q \mathfrak{sl}(n+1)$ . (For any  $a$  the elements  $z_{aj}$  form a standard basis of the space of the vector representation, while the elements  $\varepsilon_j z_{aj}^*$  form a standard basis of the space of the covector one (cf. [5]).)

Note that for  $N = 1$  the resulting  $U_q \mathfrak{su}(n, 1)$ -module  $*$ -algebra is isomorphic to  $\mathbf{C}_{q,q^{-1}}^{n+1}$ .

We will identify the algebras  $F_1^{\text{op}} = \mathbf{C} [C^{2n+1}]_q^{\text{op}}$  and  $F_2 = (\oplus_{i=1}^N \mathbf{C}^{n+1})_{q,q^{-1}}$  with their images under the canonical embeddings into  $F_2 \otimes F_1^{\text{op}}$ . Following (2.12), (3.7), introduce an anti-linear involution in  $F_2 \otimes F_1^{\text{op}}$  by

$$z_{ij} \mapsto z_{ij}^*, \quad \zeta_j \mapsto q^{2(j-n)} \zeta_j^*$$

The following proposition is a straightforward consequence of the definitions.

**Proposition 6.1.** *The elements*

$$K_i = z_{i0} \zeta_0^* - \sum_{j=1}^n z_{ij} \zeta_j^*$$

of  $F_2 \otimes F_1^{\text{op}}$  are intertwining kernels, and we have that

$$\begin{aligned} K_i K_j &= q K_j K_i \quad (i < j), \quad K_i K_j^* = q K_j^* K_i \quad (i \neq j), \\ K_i K_i^* &= q^2 K_i^* K_i \end{aligned}$$

**Corollary 6.2.** *The Poisson kernels*

$$P_i = q^{-2n} K_i K_i^*$$

are intertwining for any  $1 \leq i \leq n$ . Moreover, we have that for any  $i, j$ ,

$$P_i P_j = P_j P_i, \quad P_i^* = P_i$$

As was explained in Section 4, the powers of the Poisson kernel are generating functions for some polynomials of hypergeometric form. This allows to consider the intertwining kernels  $\prod_{j=1}^N P_j^{l_j}$  ( $l_j \in \mathbf{Z}_+$ ) as generating functions for elements of the algebra  $F_2$  which generalize classical orthogonal polynomials.

Repeating the considerations of Section 4, one can get rid of the restriction  $l_1 \in \mathbf{Z}_+$  for one of the powers of the Poisson kernels. This enables us to incorporate the case  $\sum_{j=1}^n l_j = -n$  and obtain some invariants in  $(\oplus_{i=1}^N \mathbf{C}^{n+1})_{q, q^{-1}}$  by integrating the kernel  $\prod_{j=1}^N P_j^{l_j}$ .

This approach considered in the limit case  $q = 1$  is analogous to the approach of V.A.Vasiliev, I.M.Gelfand, and A.B.Zelevinsky to generalized hypergeometric functions (cf. [4]).

## Acknowledgments.

I want to express my gratitude to V.G.Drinfeld for fruitful discussions of the role of intertwining operators in representation theory and harmonic analysis.

## References

- [1] Askey, R., Wilson, J.: *Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials*. Mem. Amer. Math. Soc. **54**, 319 (1985)
- [2] Drinfeld, V.G.: *Quantum groups*. Proc. ICM-86 (Berkeley), Amer. Math. Soc., vol. **1**, 798-820 (1987)
- [3] Faddeev, L.D., Reshetikhin, N.Yu., Takhtajan, L.A.: *Quantization of Lie groups and Lie algebras*. Leningrad Math. J. **1**, 193-226 (1990)
- [4] Gelfand, I.M., Vasiliev, V.A., Zelevinsky, A.V.: *Generalized hypergeometric functions on complex Grassmannians*. Funkts. Anal. i ego Pril. **21:1**, 23-38 (1987)
- [5] Jimbo, M.: *Quantum R-matrix associated to the generalized Toda system: an algebraic approach*. Lect. Notes in Phys. **246**, Springer-Verlag, Berlin and New York, 335-361 (1986)
- [6] Soibelman, Y.S., Vaksman, L.L.: *On some problems in the theory of quantum groups*. Adv. in Soviet Math. **9**, 3-55 (1992)
- [7] Soibelman, Y.S., Vaksman, L.L.: *Algebra of functions on the quantized group  $SU(2)$* . Funkts. Anal. i ego pril. **22:3**, 1-14 (1988)

- [8] Vaksman, L.L.: *q-analogues of the Clebsch-Gordan coefficients and algebra of functions on the quantum group  $SU(2)$* . Dokl. Akad. Nauk **306:2**, 269-271 (1989)